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Periodic Method of Characteristics

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A modification to the method of characteristics (MOC) is described for solving a system of two first order hyperbolic partial differential equations possessing periodic solutions. The condition of the physical system is specified by two boundary conditions at a single spatial location. The system's periodicity is incorporated into the numerical scheme to generate an additional set of boundary conditions at $t = 0$ and $t = 2\pi$ as the solution proceeds forward in space rather than in time as in the usual MOC. No downstream boundary conditions and no initial conditions are permitted. The number of computations is minimized since only one cycle is calculated. The validity of this new approach is illustrated by an example from cardiovascular fluid dynamics for which the exact solution is known.

INTRODUCTION

The method of characteristics (MOC) is a powerful numerical approach to the solution of hyperbolic partial differential equations. Detailed mathematical discussions of the MOC as well as applications to compressible fluid flow are covered in many texts [5, 6, 9]. The application of the MOC to hydraulic transients is discussed in [13].

More recently the MOC has been used in the study of cardiovascular fluid dynamics [1–4, 7, 10, 12, 14–16, and others]. Since the pressure and velocity in the cardiovascular system are each a superposition of forward and reverse waves, they inherently contain information regarding downstream conditions, even though this contribution cannot be explicitly stated. An advantage of the MOC is that reflections from the periphery are automatically absorbed into the solution because of the backward-running characteristic curve at each grid point.

In the usual procedure for applying the MOC to blood flow problems, a boundary condition, generally pressure as a function of time, is chosen at each end of the vessel segment under consideration, initial conditions are assumed for the variables, and the solution is propagated forward in time. The solution is assumed to be periodic. The initial transients gradually die out until the difference between

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two successive cycles is small enough to be neglected. The values for the last computed cycle then constitute the solution.

It is more advantageous experimentally, however, if the boundary conditions can be specified at a *single* site in the vessel in order to predict the flow and pressure at any other point. Because a periodic solution exists for the blood flow problem, an approach has been developed wherein two upstream boundary conditions are specified. The periodicity of the solution allows a set of boundary conditions at $t = 0$ and $t = 2\pi$ to be continually generated as the solution proceeds down the vessel (forward in space) rather than forward in time.

In Section 1, this new approach, termed the periodic method of characteristics (PMOC), is described for the case of two general hyperbolic differential equations possessing periodic solutions. In Section 2, the validity of the method is illustrated by a simple example from cardiovascular fluid dynamics for which the exact solution is known.

1. MOC Applied to a General Periodic System

The system to be considered is described by the two quasilinear hyperbolic partial differential equations

$$L_1 = A_1 \frac{\partial u}{\partial t} + B_1 \frac{\partial u}{\partial x} + C_1 \frac{\partial v}{\partial t} + D_1 \frac{\partial v}{\partial x} + E_1 = 0, \tag{1}$$

$$L_2 = A_2 \frac{\partial u}{\partial t} + B_2 \frac{\partial u}{\partial x} + C_2 \frac{\partial v}{\partial t} + D_2 \frac{\partial v}{\partial x} + E_2 = 0, \tag{2}$$

where u and v are the dependent variables, x and t are the independent variables, and the coefficients $A_1, \ldots, E_2$ are known functions of the dependent and independent variables, e.g., $A_1 = A_1(x, t, u, v)$. All functions are assumed to be continuous and to possess as many continuous derivatives as may be required. The coefficients are assumed to be such that nowhere does $A_1/A_2 = B_1/B_2 = C_1/C_2 = D_1/D_2$. It is further assumed throughout that the solutions to (1) and (2) for *u and v are periodic in t.* Although the remarks below apply for any coordinates x and t, it will be convenient to call x the spatial coordinate and t the temporal coordinate. We are thus dealing with the standard Cauchy problem for two independent variables, with data specified along the line $x = 0$, i.e.,

$$u(0, t) = u_0 + \sum_{n=1}^{N} u_n \sin(nt + \psi_n), \tag{3}$$

$$v(0, t) = v_0 + \sum_{n=1}^{N} v_n \sin(nt + \phi_n), \tag{4}$$

where N is the maximum harmonic considered, u_n and v_n are the magnitudes and ψ_n and ϕ_n are the phases of the n-th harmonic.

The solution space is illustrated in Fig. 1. The solution is sought between

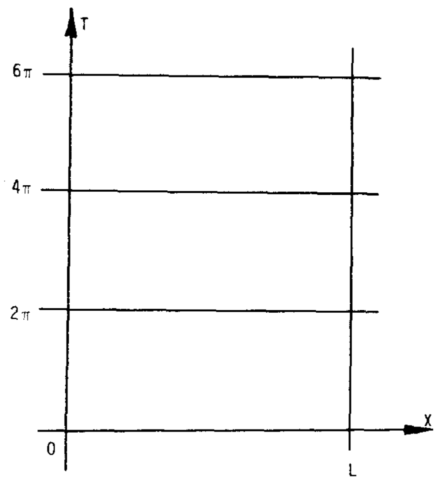

FIG. 1. Solution space.

$x = 0$ and $x = L$ and for all time. Because the solutions to (1) and (2) are periodic in t, the time can be nondimensionalized so that 2π represents the length of the period in the t direction, limiting the discussion to the solutions between $t = 0$ and $t = 2\pi$.

The system L_1 and L_2 is readily reduced to the equivalent total differential system by the method of characteristics [8, 11]:

$$\textit{forward}: \qquad A^+(du/dt) + C^+(dv/dt) + E^+ = 0, \tag{5}$$

$$\textit{backward}: \qquad A^-(du/dt) + C^-(dv/dt) + E^- = 0, \tag{6}$$

where $A^+ = A_1 + \lambda_+ A_2$, $A^- = A_1 + \lambda_- A_2$, etc., and λ_+ and λ_- are the Lagrangian multipliers corresponding to the forward and backward characteristic curves, respectively. Thus, the problem reduces to solving for the periodic solutions to (5) and (6) with characteristic directions of $dx/dt = \zeta^+$ and $dx/dt = \zeta^-$, respectively.

The particular computational method developed here utilizes the method of specified time intervals as outlined in [8]. This method is preferred over the grid of characteristics method (GOCM) because of its more orderly computational scheme. The results are presented directly in the output format needed for plotting, regardless of the output location(s) selected. The GOCM demands an additional interpolation to put the output values in a useful format. However, the GOCM eliminates interpolations in the numerical scheme itself. Figure 2 defines the grid pattern used, where the lines labeled C_+ and C_- are the forward and backward characteristic curves, respectively. The grid runs from 0 to 2π in the t direction and from 0 to some terminal value L in the x direction as shown in Fig. 1.

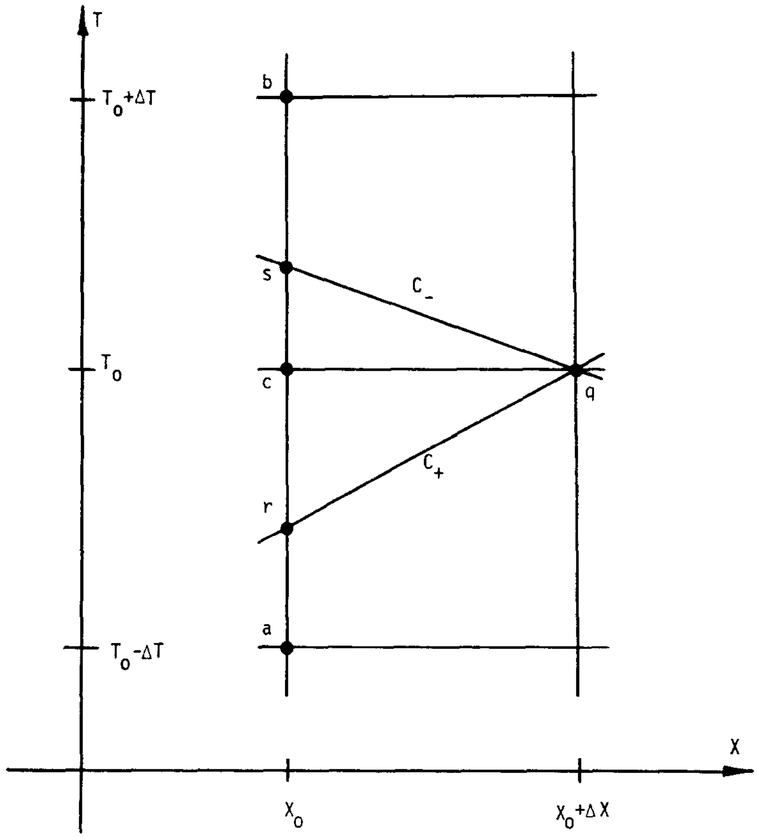

FIG. 2. Grid diagram.

The time increment is $\Delta t = 2\pi/NT$, where NT is the number of segments between $t = 0$ and $t = 2\pi$. The distance increment is $\Delta x = 1/NX$, where NX is the number of segments per unit distance. The constraint on the choice of Δx, given Δt, is that the characteristics must fall on the line segment a–b. With this constraint met, the solution exists and is unique [5, 9]. Increasing the number of grid points increases the accuracy of this solution, but this principle is subject to the law of diminishing returns on invested computer time. If a larger number of grid points in the t direction is chosen, thereby reducing Δt, the above constraint dictates that a smaller Δx must also be chosen. Thus, computation time is proportional to the square of the grid points in the t direction, an important practical factor in choosing the value for NT.

In the following equations, the subscripts refer to the points on the grid diagram of Fig. 2 where the subscripted variables are to be evaluated. The solution procedure is as follows: (1) Find t_r and t_s by propagating the characteristic curves from point q back to the line segment x_0 ; (2) Determine the values of u and v at points r and s by interpolation of the values previously known at points a, b, and c; (3) Use Eqs. (5) and (6) to give two equations for u and v. Below are listed the equations to be employed by this procedure. It will be noted that (9)–(12) are simple linear interpolations. If u and v change rapidly from node to node, quadratic or trigonometric interpolations should be used to give greater accuracy

$$t_r = t_c - \Delta x/\zeta_c^+, \tag{7}$$

$$t_s = t_c - \Delta x/\zeta_c^-, \tag{8}$$

$$u_r = [(t_c - t_r)/\Delta t]\, u_a + [1 - (t_c - t_r)/\Delta t]\, u_c\,, \tag{9}$$

$$v_r = [(t_c - t_r)/\Delta t]\, v_a + [1 - (t_c - t_r)/\Delta t]\, v_c\,, \tag{10}$$

$$u_s = -[(t_c - t_s)/\Delta t]\, u_b + [1 + (t_c - t_s)/\Delta t]\, u_c\,, \tag{11}$$

$$v_s = -[(t_c - t_s)/\Delta t]\, v_b + [1 + (t_c - t_s)/\Delta t]\, v_c\,, \tag{12}$$

$$A_r^+u_q + C_r^+v_q = A_r^+u_r + C_r^+v_r + (t_r - t_c)\, E_r^+ \equiv K_r\,, \tag{13}$$

$$A_s^-u_q + C_s^-v_q = A_s^-u_s + C_s^-v_s + (t_s - t_c)\, E_s^- \equiv K_s\,, \tag{14}$$

$$u_q = (K_rC_s^- - K_sC_r^+)/(A_r^+C_s^- - A_s^-C_r^+), \tag{15}$$

$$v_q = (K_sA_r^+ - K_rA_s^-)/(A_r^+C_s^- - A_s^-C_r^+). \tag{16}$$

A single iteration may be sufficiently accurate to define u_q and v_q . As a check, t_r and t_s can be recalculated using the average values of ζ^+ between r and q and ζ^- between s and q, respectively. If the new t_r and t_s fall within $\epsilon\, \Delta t$ of the previous values, where ϵ is a small convergence factor, then the calculations for grid point q can be considered complete. If not, more iterations can be performed using average values of the variables and coefficients between r and q, and s and q, until the convergence criterion is met.

As can be seen from Fig. 2, three "old" grid points are needed to calculate u and v at each new point q. But at the end points of the line segment $x = x_o + \Delta x$, only two old grid points are available. Thus, two fewer grid points can be calculated for each successive value of x.

The periodicity of the solution is invoked to resolve this difficulty. As illustrated in Fig. 3, the calculation grid consists of $NT + 2$ nodes in the t direction, with node $IT = 1$ corresponding to $t = 0$ and $IT = NT + 1$ to $t = 2\pi$. The periodicity condition says that the values at $t = 0$ and $t = 2\pi$ (or $IT = 1$ and $IT = NT + 1$) must be equal and that the values at $IT = 2$ and $IT = NT + 2$ must also be equal. Using the line segment at $x = x_0$ with $NT + 2$ nodes, NT nodes (from $IT = 2$ to $IT = NT + 1$) are calculated at $x = x_0 + \Delta x$. The values for $IT = 1$ and $IT = NT + 2$ at $x = x_0 + \Delta x$ are set to the newly calculated values at

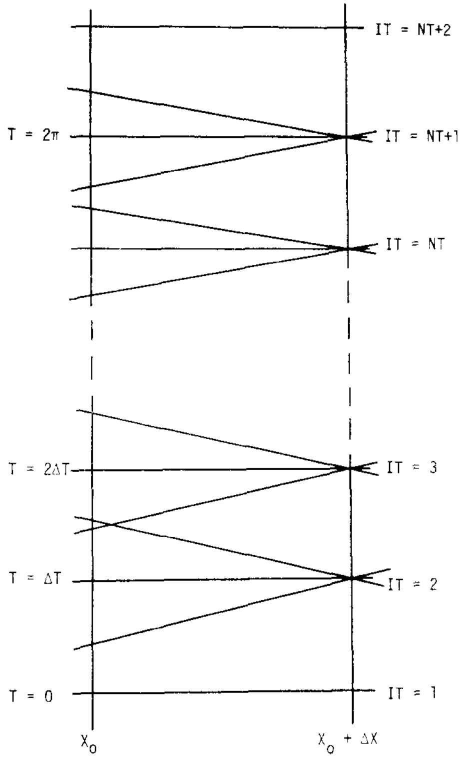

FIG. 3. Periodicity condition.

$IT = NT + 1$ and $IT = 2$, respectively. Thus, at each line segment x, each of the $NT + 2$ nodes *can* be evaluated, before proceeding to the next x position. In effect, the periodicity condition allows the generation of an additional set of boundary conditions while the computation is in progress.

Only one cycle is calculated for each value of x. If L is the termination site, and NX the number of divisions between $x = 0$ and $x = 1$, then the total number of nodes which must be evaluated is $L(NT + 2)\,NX$.

The formulation of the numerical scheme does not allow the specification of downstream boundary conditions at $x = L$ and initial conditions at $t = 0$. Thus, if this method is used for solving fluid flow problems, for example, the flow must be isentropic. Otherwise, the problem would not reduce to only two partial differential equations.

2. Comparison with an Exact Solution

The validity of the above procedure may be illustrated by a problem in cardiovascular fluid dynamics. One-dimensional analysis of blood flow in an artery can be simplified by eliminating the friction forces and any outflow through the vessel walls, neglecting the convective terms in the equation of motion, assuming the pulse wave speed is constant, and considering the artery to be untapered [14]. The two conservation equations become

Continuity: $$\Omega(\partial P/\partial t) + (\partial u/\partial x) = 0, \tag{17}$$

Momentum: $$\Omega(\partial u/\partial t) + (\partial P/\partial x) = 0, \tag{18}$$

where P = pressure, u = velocity, t = time, x = axial coordinate, and $\Omega = \omega R_0/c$ is a nondimensional parameter where ω = circular frequency of first harmonic, R_o = mean radius, and c = pulse wave speed. All other variables in (17) and (18) have been nondimensionalized. These equations reduce to the classical linear wave equation

$$(\partial^2 u/\partial x^2) - \Omega^2(\partial^2 u/\partial t^2) = 0, \tag{19}$$

which has a solution given by the familiar d'Alembert formula

$$u(x, t) = \frac{1}{2}\,[f(t + \Omega x) + f(t - \Omega x)] + \frac{1}{2\Omega}\int_{t-\Omega x}^{t+\Omega x} g(z)\,dz. \tag{20}$$

The functions f and g are obtained from the boundary conditions for u and P at $x = 0$:

$$u(0, t) = u_0 + \sum_{n=1}^{N} u_n \sin(nt + \psi_n) \equiv f(t), \tag{21}$$

$$P(0, t) = P_0 + \sum_{n=1}^{N} P_n \sin(nt + \phi_n). \tag{22}$$

To obtain g, (22) is differentiated and combined with (17):

$$\frac{\partial u(0, t)}{\partial x} = -\Omega\,\frac{\partial P(0, t)}{\partial t} = -\Omega \sum_{n=1}^{N} P_n n \cos(nt + \phi_n) \equiv g(t). \tag{23}$$

An analogous solution, of course, can be written for the pressure P.

The system of equations given by (17) and (18) was programmed by the procedure outlined in the previous section and compared with the exact solutions for u and P given by (20)–(23) and their counterpart for P. For input conditions at $x = 0$, pressure and velocity data from the aorta of a healthy dog was used. The parameters of interest are $N = 10$, $\epsilon = 0.001$, $R_o = 0.945$ cm, $c = 650$ cm/sec, and $\omega = 16.23$ rad/sec. A single iteration at each node gave the accuracy specified by ϵ. Figure 4 shows the comparison of the new numerical method with the exact

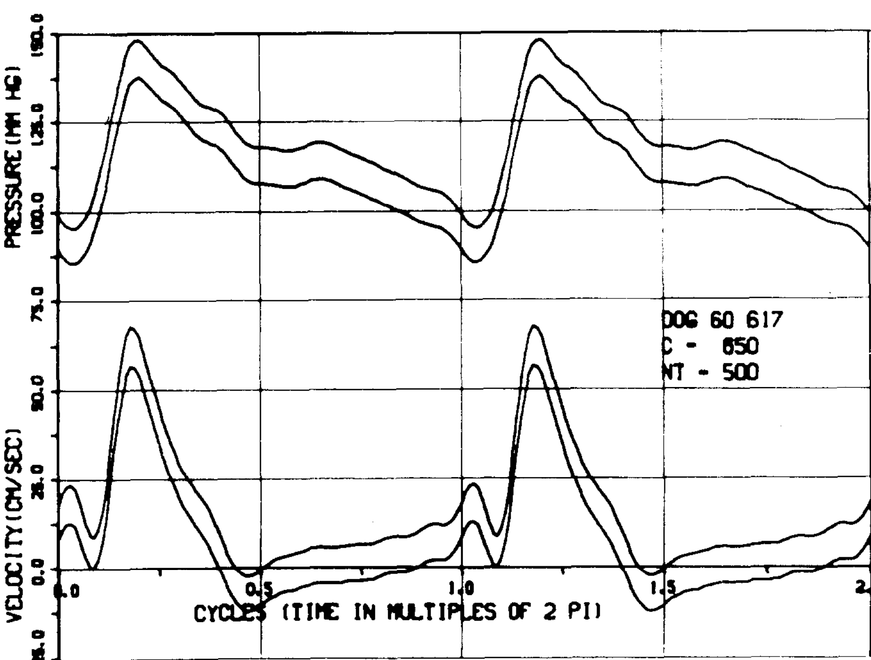

FIG. 4. Aortic waveform development comparison of numerical and exact solution with $NT = 500$ and $NX = 10$.

solution at an axial location of 15 radii from the input site. The pressure and velocity as calculated by the PMOC are lowered by 10 mm Hg or 10 cm/sec, respectively, from the linear curves, for readability. The PMOC used the grid parameters of $NT = 500$ and $NX = 10$. As can be seen, the numerical method compares quite favorably with the exact solution. Reducing NT to 100 and NX to 2 introduces discrepancies of about 5 cm/sec in the velocity curve preceding the rise and again at peak velocity as shown in Fig. 5; the pressure wave appears relatively unaffected by the change in grid size. Central processor time on the Control Data 6400 Computer was 4 sec for the case with $NT = 100$ and 6 sec for $NT = 500$.

This problem is presented only to illustrate the validity of the PMOC. The problems of real interest in cardiovascular dynamics involve all the nonlinearities that were neglected in setting up (17) and (18), e.g., taper, friction, and convection. The need for accurate experimental data used for input values at $x = 0$ is obvious.

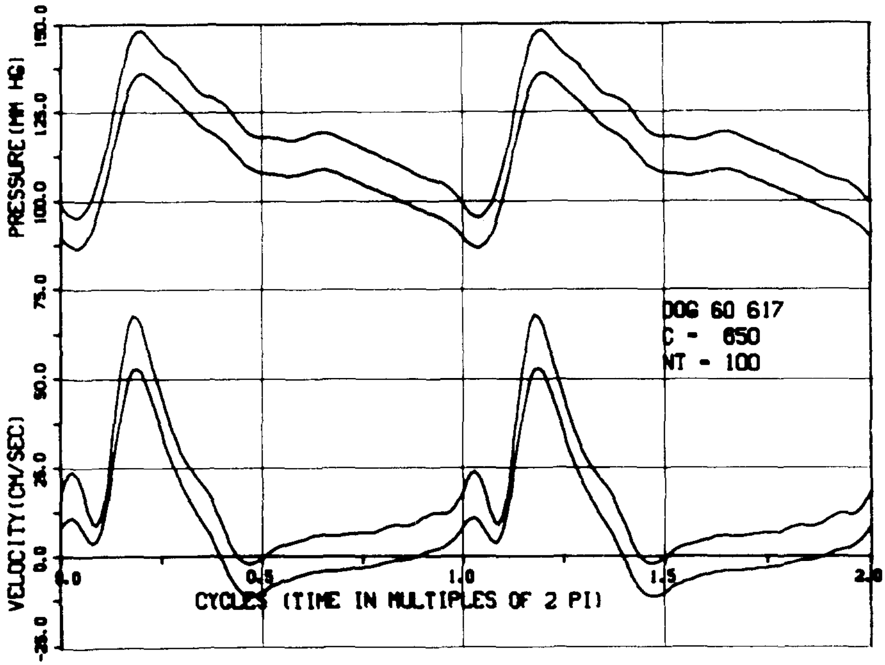

FIG. 5. Aortic waveform development comparison of numerical and exact solution with $NT = 100$ and $NX = 2$.

CONCLUSIONS

A modification to the method of characteristics makes it possible to find the complete solution to a periodic system described by two hyperbolic partial differential equations from knowledge of the condition of the system for a single cycle at a single physical point. The modification permits no downstream boundary conditions at $x = L$ and no initial conditions at $t = 0$. This new periodic method of characteristics is computationally efficient since only one cycle must be calculated.

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